Exercise 7.3.4

Find the general solutions to the following ODEs. Write the solutions in forms that are entirely real (i.e., that contain no complex quantities).

$$y'' + 2y' + 2y = 0.$$

Solution

Because this is a linear homogeneous ODE and all the coefficients on the left side are constant, the solutions for it are of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these formulas into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 2 = 0$$

Solve for r.

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$r = \{-1 - i, -1 + i\}$$

Two solutions to the ODE are $y = e^{(-1-i)t}$ and $y = e^{(-1+i)t}$. By the principle of superposition, the general solution is a linear combination of these two. Therefore,

$$\begin{aligned}
y(t) &= C_1 e^{(-1-i)t} + C_2 e^{(-1+i)t} \\
&= C_1 e^{-t-it} + C_2 e^{-t+it} \\
&= C_1 e^{-t} e^{-it} + C_2 e^{-t} e^{it} \\
&= C_1 e^{-t}[\cos(-t) + i\sin(-t)] + C_2 e^{-t}[\cos(t) + i\sin(t)] \\
&= C_1 e^{-t}(\cos t - i\sin t) + C_2 e^{-t}(\cos t + i\sin t) \\
&= C_1 e^{-t}\cos t - iC_1 e^{-t}\sin t + C_2 e^{-t}\cos t + iC_2 e^{-t}\sin t \\
&= (C_1 + C_2)e^{-t}\cos t + (-iC_1 + iC_2)e^{-t}\sin t \\
&= C_3 e^{-t}\cos t + C_4 e^{-t}\sin t \\
&= e^{-t}(C_3\cos t + C_4\sin t).
\end{aligned}$$